Lecture 1   Fibre bundles

We assume everyone is familiar with the notions of differentiable manifolds. I may include scholia on topics which may not be familiar. All manifolds will be smooth, paracompact, Hausdorff & finite-dimensional.

A fibre bundle generalises the notion of a product. The fundamental property is that of local triviality, which says that a fibre bundle is locally a product.

1. Definition A fibre bundle consists of a smooth surjection $\pi : E \to M$ between manifolds E (the total space) and M (the base) and such that for every $a \in M$ there exists a neighbourhood $U \ni a$ and a diffeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ (a local trivialisation), for some manifold F (the typical fibre) such that the following triangle commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "pr_1"] \\
U &
\end{tikzcd}$$

Notation We often write $F \to E \xrightarrow{\pi} M$. If we can take $U = M$, so that $\varphi : E \xrightarrow{\cong} M \times F$, we say E is a trivial bundle.

Now suppose that (U, φ) and (V, ψ) are local trivialisations with $U \cap V \neq \emptyset$. Then we have two ways to view $\pi^{-1}(U \cap V)$ as a product:

$$\begin{tikzcd}
U \cap V \times F \arrow[dr, "pr_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\psi"'] \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \cap V \times F \arrow[dl, "pr_1"] \\
& U \times V &
\end{tikzcd}$$

and hence

$$\psi \circ \varphi^{-1} : U \cap V \times F \longrightarrow U \cap V \times F$$
$$(a, p) \longmapsto (a, \Phi(a,p))$$

where $\Phi(a, -) : F \longrightarrow F$ is a diffeomorphism.

In other words, it defines a transition function

$$g : U \cap V \longrightarrow \mathrm{Diff}(F)$$

2. Definition Let $F \to E \xrightarrow{\pi} M$. A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ (A some indexing set) where $\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ is a local trivialisation and $M = \bigcup_{\alpha \in A} U_\alpha$ is called a trivialising atlas for $E \xrightarrow{\pi} M$. Let's introduce the notation $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, et cetera. Then let $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Diff}(F)$ be the transition functions defined by $\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$, $\varphi_\alpha \circ \varphi_\beta^{-1}(a, p) = (a, g_{\alpha\beta}(a) p))$.

Exercise 1. Show that the transition functions $\{g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(F)\}$ obey the cocycle conditions:

$$g_{\alpha\alpha}(a) = \mathrm{id}_F \quad \text{for all } a \in U_\alpha$$

$$g_{\alpha\beta}(a)\, g_{\beta\alpha}(a) = \mathrm{id}_F \quad \text{for all } a \in U_{\alpha\beta}$$

$$g_{\alpha\beta}(a)\, g_{\beta\gamma}(a) = g_{\alpha\gamma}(a) \quad \text{for all } a \in U_{\alpha\beta\gamma}$$

3. Definition Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} N$ be fibre bundles. A bundle map is a pair (Φ, φ) of smooth maps $\Phi : E \to E'$ and $\varphi : M \longrightarrow N$ such that the following square commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd} \qquad \text{i.e.: } \pi' \circ \Phi = \varphi \circ \pi$$

Since π is surjective, φ is uniquely determined by Φ. Φ is said to cover φ.
Notice that Φ is fibre-preserving.

An important special case is where $M = N$ and Φ covers the identity:

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

so Φ maps the fibres E_a to E'_a for all $a \in M$.

4. Definition Let $f : M \to N$ be smooth and let $E \xrightarrow{\pi_E} N$ be a fibre bundle. Then we can define the pull-back bundle $f^*E \to M$ as the "categorical pull-back" of $f : M \to N$ and $\pi_E : E \to N$:
$f^*E := \{(a, e) \in M \times E \mid \pi_E(e) = f(a)\} \subset M \times E$. Restricting the canonical projections $pr_1 : M \times E \to M$ and $pr_2 : M \times E \to E$ to f^*E we get maps $\pi : f^*E \to M$ and $\Phi : f^*E \to E$ making the following square commute:

$$\begin{tikzcd} f^*E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi_E"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Let $a \in M$ and let (V, Ψ) be a local trivialisation for $E \to N$ with $f(a) \in V$. Then $(f^{-1}(V), \varphi)$, where $\varphi : \pi^{-1}(f^{-1}(V)) \to f^{-1}(V) \times F$ is defined by $\varphi(b, e) = (b, pr_2(\Psi(e)))$, is a local trivialisation for $f^*E \to M$.
This shows $f^*E \to M$ is a fibre bundle and (Φ, f) is a bundle map. Notice that $(f^*E)_a = E_{f(a)}$.

5. **Definition** A section of a fibre bundle $F \to E \xrightarrow{\pi} M$ is a smooth map $s: M \to E$ such that $\pi \circ s = \mathrm{id}_M$. In other words, it is a smooth assignment to every $a \in M$ of a point in the fibre $E_a = \pi^{-1}(a)$. Sections may not exist, but if the fibre bundle is trivial, then any smooth map $\sigma: M \to F$ defines a section by $s(a) = (a, \sigma(a))$. Since fibre bundles are locally trivial, they always admit local sections $s_\alpha: U_\alpha \to \pi^{-1}(U_\alpha)$ via local smooth maps $\sigma_\alpha: U_\alpha \to F$ as follows:

$$\begin{tikzcd} U_\alpha \arrow[r, dashed, "s_\alpha"] \arrow[d, "\mathrm{id}\times\sigma_\alpha"'] & \pi^{-1}(U_\alpha) \\ U_\alpha \times F \arrow[ur, "\varphi_\alpha^{-1}"'] & \end{tikzcd}$$

A section $s: N \to E$ through $E \xrightarrow{\pi_N} N$ induces a section $f^*s: M \to f^*E$ via $(f^*s)(a) = (a, s(f(a))) \in f^*E$.

6. **Definition** Consider $F \to E \xrightarrow{\pi} M$. Then the fibres $\pi^{-1}(a) \subset E$ are submanifolds of E. The tangent space to the fibre $E_a = \pi^{-1}(a)$ at $e \in \pi^{-1}(a)$ is the kernel of $(\pi_*)_e : T_eE \to T_aM$. It is called the vertical subspace of T_eE and denoted $\mathcal{V}_e$. In the absence of additional structure, there is no preferred complementary subspace of T_eE. A connection on $E \to M$ is a smooth choice of complementary subspace: $\mathcal{H}_e \subset T_eE$ s.t. $T_eE = \mathcal{V}_p \oplus \mathcal{H}_p$. That is, a distribution $\mathcal{H} \subset TE$. Notice that $(\pi_*)_e\big|_{\mathcal{H}_e} : \mathcal{H}_e \xrightarrow{\cong} T_{\pi(e)}M$ so $\mathcal{H}$ gives a privileged way to lift tangent vectors (& curves) from M to E.

Given a distribution $\mathcal{H} \subset TE$ one can ask whether it is integrable (in the sense of Frobenius): ie: is E foliated by submanifolds whose tangent spaces agree with $\mathcal{H}$? (eg: if $E \to M$ is trivial, we can take $\mathcal{H}$ to be the tangent spaces to M.) We shall see that the obstruction to the integrability of $\mathcal{H} \subset TE$ can be interpreted as the 'curvature' of the connection.

whether sections of $\mathcal{H}$ close under the Lie bracket

So far this discussion has been quite general. We will now specialise to different kinds of fibre bundles, by considering special kinds of typical fibres: Lie groups, homogeneous spaces, vector spaces, ... and demanding that the local trivialisations respect some structure.